Remark on the effective range approach to analyzing the $f_0 - a_0$ mesons

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Abstract

An over simplified analysis of issues related to a resonance close to threshold may lead to misleading results. We clarify some subtle points, in particular the relation between the Breit-Wigner and effective range approaches to the $f_0 - a_0$ mesons.

The f_0 and a_0 scalar mesons present a well-known puzzle for which several interesting, albeit controvertial proposals have been made ranging from quark-antiquark makeup to four-quark and to $K\bar{K}$ molecular structure. The question of how to distinguish between different hypotheses for the $f_0 - a_0$ nature has been discussed in a great number of papers and it is not our goal to review all these suggestions. Long ago it was proposed to use the sign and the value of the $K\bar{K}$ effective range parameter as a signature of the $f_0 - a_0$ makeup [1]. Namely, it was suggested that the large and negative value of r_e corresponds to the case of a large admixture of the bare quark state while a small (positive or negative) r_e is a sign that the resonance contains a large mesonic component [1]. The question of whether it is possible to judge the nature of the $f_0 - a_0$ mesons having at hand the values of the low-energy $K\bar{K}$ parameters has been revisited in a recent arXive submission hep-ph/0308129 [2]. In our opinion some remarks on this, presented below, significantly qualify the answer to question regarding the distinguishability of the $f_0 - a_0$ makeup on the basis of their phenomenological parameters. In this note we do not discuss the question of to what extent the value and the sign of the effective range are sound signatures of resonance makeup. What

will be shown is that scrutiny of the effective range calculations presented in [2] results in serious doubts about their reliability.

Equation (20) of [2] for the $K\bar{K}$ effective range reads

$$r_e = -\frac{4}{m\bar{g}_{K\bar{K}}}. \quad (1)$$

Here $m = (m_{K^+} m_{K^0})/2, \bar{g}_{K\bar{K}} = g^2_{K\bar{K}}/8\pi M^2$, where M is the mass of the resonance (a_0 or f_0), and $g^2_{K\bar{K}}$ is the standard coupling constant connected to the resonance width via $\Gamma_{K\bar{K}} = g^2_{K\bar{K}}(8\pi M^2)^{-1}k$, where k is the c.m. momentum. For definiteness in what follows we consider only the f_0- case.

The value of the coupling constant $\bar{g}_{KK}$ suffers a large ambiguity, due to the variety of experimental data and the lack of consensus in its definition. Hence the results for the effective range presented in Table 2 of [2] and calculated according to (1) span from -0.56 fm to -1.22 fm. One may add another line to the Table 2 of [2] using the value of $\bar{g}_{K\bar{K}}$ presented in [3] (set D of Table 1 in [3]); then equation (1) yields $r_e = -4.79$ fm. Following the arguments of [1, 2] one should conclude that the value of $r_e = -0.56$ fm corresponds to a large admixture of the $K\bar{K}$ component while $r_e = -4.79$ fm is evidence for a dominant quark nature of the f_0– meson. We do not want to discuss this obvious contradiction. Our next step is to replace equation (1) by the one which correctly takes into account the interplay of a quark state and a hadronic threshold. We shall show that, with $\bar{g}_{K\bar{K}}$ constants the same as in [2], the value $r_e = -0.56$ fm is replaced by $r_e \simeq +0.8$ fm, while instead of $r_e = -4.79$ fm one gets $r_e \simeq -3.5$ fm.

The problem may be approached in different ways. In particular, a correct treatment is presented in [4], where one finds a simple model for the interplay of quark and hadronic channels in the f_0-meson. Here we follow an alternative approach based on the formalism developed in [5]-[8]. Consider a three-channel system with two hadronic channels denoted by π (the $\pi\pi$) and K (the $K\bar{K}$) and a quark channel denoted by q ($q\bar{q}$ or $q^2\bar{q}^2$). As we shall see in a moment, the $\pi\pi$ channel plays no dynamical role in the model and therefore relativistic treatment which is formally needed for the $\pi\pi$ system would not change anything. The quark channel is parametrized by the position of the bare level E_n and the communication potentials V_{qi}, $i = \pi, K$. We assume that there is no residual potential interaction between the two hadrons ($\pi\pi$ or $K\bar{K}$). The generalization of this model to the case of several levels in the quark channel and the hadronic interaction may be found in [7, 8]. The

exact equation for the S-matrix in the channel K reads [7, 8]

$$S_K(E) = \frac{E - E_n + \langle q|V_{q\pi}G_\pi^{(+)}V_{\pi q}|q\rangle + \langle q|V_{qK}G_K^{(-)}V_{Kq}|q\rangle}{E - E_n + \langle q|V_{q\pi}G_\pi^{(+)}V_{\pi q}|q\rangle + \langle q|V_{qK}G_K^{(+)}V_{Kq}|q\rangle}, \tag{2}$$

where the indices $(+)$ and $(-)$ in (2) correspond to the choice of the boundary conditions $\pm i0$ in the Green's functions of the corresponding channels. Since we are interested in the phenomena taking place in the energy interval of the order of a few tens of MeV around the $K\bar{K}$ threshold, we may neglect the energy dependence of the $\pi\pi$ matrix element and parametrize it in a standard way

$$\langle q|V_{q\pi}G_\pi^{(+)}V_{\pi q}|q\rangle = -\varepsilon_{\pi\pi} + \frac{i}{2}\Gamma_{\pi\pi}, \tag{3}$$

where $\varepsilon_{\pi\pi}$ is the hadronic shift of the level and $\Gamma_{\pi\pi}$ is the width of f_0-meson into $\pi\pi$. We shall include the constant shift $\varepsilon_{\pi\pi}$ into the energy of the level E_n. The energy dependence of a similar matrix element for the $K\bar{K}$ channel is crucial. Again as in (3) we represent the matrix element for the $K\bar{K}$ channel as a sum of Hermitian and anti-Hermitian parts depending upon the $K\bar{K}$ c.m. momentum k

$$\langle q|V_{qK}G_K^{(\pm)}V_{Kq}|q\rangle = -\varepsilon_{K\bar{K}}(k) \pm \frac{i}{2}\Gamma_{K\bar{K}}(k). \tag{4}$$

Equations (2)-(4) yield the following T-matrix for the $K\bar{K}$ channel

$$T_K = (2\pi^2 mk)^{-1}\frac{\Gamma_{K\bar{K}}(k)/2}{E - E_n + \frac{i}{2}\Gamma_{\pi\pi} - \varepsilon_{K\bar{K}}(k) + \frac{i}{2}\Gamma_{K\bar{K}}(k)}. \tag{5}$$

At this point it is tempting to parametrize $\Gamma_{K\bar{K}}(k)$ as $\Gamma_{K\bar{K}}(k) = \bar{g}_{K\bar{K}}k$ and to include $\varepsilon_{K\bar{K}}(k)$ into E_n (as it was previously done with $\varepsilon_{\pi\pi}$). Then one arrives at Eq. (17) of [2], namely

$$f_K = -\frac{\bar{g}_{K\bar{K}}/2}{\frac{k^2}{m} - E_n + \frac{i}{2}\Gamma_{\pi\pi} + \frac{i}{2}\bar{g}_{K\bar{K}}k}, \tag{6}$$

which immediately leads to the expression (1) for the effective range r_e.

What was done wrongfully in passing from (5) to (6)? The answer is clear: the term proportional to k^2 stemming from $\varepsilon_{K\bar{K}}(k)$ was omitted. This omission is mentioned in [2] without further discussion. When included, this term adds to the expression (1) the contribution which is of the same order

as (1) but of the opposite sign. To see this let us return to Eq. (4) and write the matrix element explicitly

$$-\varepsilon_{K\bar{K}}(k)+\frac{i}{2}\Gamma_{K\bar{K}}(k)=\int d\mathbf{p}d\mathbf{p}'\langle q|V_{qK}|\mathbf{p}\rangle\frac{\delta(\mathbf{p}-\mathbf{p}')}{\frac{p^2}{m}-E(k)-i0}\langle\mathbf{p}'|V_{Kq}|q\rangle=$$

$$=4\pi m\int dpp^2\frac{|\langle q|V_{qK}|\mathbf{p}\rangle|^2}{p^2-k^2-i0}. \tag{7}$$

The result (1) of [2] for r_e corresponds to the assumption that the principal value of the integral (7) does not depend on k^2. Then the constant shift $\varepsilon_{K\bar{K}}$ may be absorbed into E_n as it was done with $\varepsilon_{\pi\pi}$.

Let us demonstrate that this assumption breaks down. To make (7) easily tractable consider the most widely used model for the communication potential $V(r)$, namely

$$\langle r|V_{Kq}|q\rangle=\gamma^{1/2}\frac{\delta(r-b)}{b\sqrt{4\pi}}, \tag{8}$$

where γ is a constant with the dimension of mass, and b is the range at which the transitions between the channels effectively occur. The formfactor in momentum space corresponding to the transition potential (8) reads

$$\langle q|V_{qK}|k\rangle=\int d\mathbf{r}\langle q|V_{qK}|\mathbf{r}\rangle\langle\mathbf{r}|k\rangle=\frac{\gamma^{1/2}}{\pi\sqrt{2}}\frac{\sin kb}{k}. \tag{9}$$

Therefore we are dealing with a smooth transition formfactor in momentum space with the range $k\sim\pi/2b$, i.e. $b\simeq\pi/2\beta$ in the notations of Ref. [2]. Substituting (9) into the integral (7), one gets

$$-\varepsilon_{K\bar{K}}(k)+\frac{i}{2}\Gamma_{K\bar{K}}(k)=\gamma\frac{m}{k}e^{ikb}\sin kb\simeq$$

$$\simeq\gamma mb(1-\frac{2}{3}k^2b^2)+im\gamma b^2k. \tag{10}$$

If, following [2], we use the parametrization $\Gamma_{K\bar{K}}(k)=\bar{g}_{K\bar{K}}k$, we have to identify

$$\bar{g}_{K\bar{K}}=2\gamma mb^2. \tag{11}$$

From (5) and (10) the effective range is easily calculated to be

$$r_e'=-\frac{4}{m\bar{g}_{K\bar{K}}}+\frac{4}{3}b=r_e+\frac{4}{3}b, \tag{12}$$

where r_e is given by Eq. (1) which is identical to Eq. (20) of [2]. The authors of Ref.[2] point out that with Eq. (1) it is impossible to reproduce the deuteron-like situation with positive effective range. Eq.(12) is free of this deficiency.

Now we return to Eq.(8) and discuss the physics behind it. The formal multichannel scattering theory with the communication potential (8) was developed in Ref. [6]. Less rigorous approach was followed in [7] and in [8]. Historically the use of the boundary condition (8) probably goes back to a seminal paper by C.Bloch [9]. Numerous calculations of different quark-hadron systems based on (8) were performed - see e.g. [10, 11, 12]. A very important observation was done in [13], namely that the model with the δ -function transition potential (8) is equivalent to the Jaffe- Low P-matrix [14]. Therefore we may use the well-known P-matrix recipes to estimate the range b and the coupling constant γ. For meson-meson system this yields [4, 14, 15, 16]

$$b \simeq 1.4R, \quad R \simeq 5M^{1/3}\text{GeV}^{-1}, \tag{13}$$

where M is the mass of the bare quark state, $M \simeq 1$ GeV for f_0 -meson, i.e. $R \simeq 1$ fm. The absolute lower bound on b is $b_{min} = 0.4R \simeq 0.4$ fm [4]. As it was mentioned after (9) the corresponding range of the transition formfactor in momentum space is $k \equiv \beta \simeq \pi/2b_{min} \simeq 800$ MeV. Obviously the nonrelativistic approach used in [2] and in the present work completely breaks at such values of the relative $K\bar{K}$ momenta. Therefore a conservative estimate of b is $b \simeq 1fm \simeq 5$ GeV^{-1}. According to (12) it means that the values of the effective range are 1.3 fm larger than the results presented in Table 2 of [2]. The values of r_e listed in this Table range from -0.56 fm to -1.22 fm, and hence we are dealing with $\rangle$100% "correction".

The connection between the formfactor (8) and P-matrix allows to estimate the coupling constant γ. It is related to the residue $\lambda_{K\bar{K}}$ of the P-matrix with respect to the $K\bar{K}$ channel via $\Gamma = \lambda_{K\bar{K}}/m$ [17]. This residue is known very approximately, $\lambda_{K\bar{K}} \simeq 0.02$ GeV^2 [16], so that $\gamma \simeq 0.04$ GeV. According to (11) this corresponds to $\bar{g}_{K\bar{K}} \simeq 1$, but this result relies on the rather uncertain value of $\lambda_{K\bar{K}}$. Once more we see that the determination of the coupling constant $g^2_{K\bar{K}}(f_0)$ is a problem still waiting for its solution.

The situation with P-matrix parameters is much more clear in the NN sector. As an example consider the set of parameters for $^3S_1$ state from Ref. [18], namely $\tau = 2m_N^2\gamma = 0.4$ GeV^3, $b = 7.16$ GeV^{-1}. According to (11) and (1) one gets $r_e = -4/\tau b^2 = -0.039$ fm, while the correct Eq. (12) yields

$r'_e = 1.87$ fm, which is close to the standard result $r_e = 1.75$ fm [19].

We also note that from the general expression (2) for the S- matrix one can calculate the relative weights of quark and hadron components in the physical f_0 -meson. The corresponding equation may be found in Ref.[8] (Eq. (19)) and in Ref. [20].

One may ask a question to what extent are the effective range calculations presented above model dependent. In particular, whether the second term in Eq. (12) is really important, or it contributes a minor correction in line with the statement of Ref. [2]. To clear out possible doubts let us turn to the model independent calculation of the one-loop scalar propagator, or 1PI two-point function [3,21,22]. Using the standard dimensional regularization of the loop diagram we find the following expression for the finite part of the inverse propagator

$$D(s) = s - M^2 + \Sigma(s), \tag{14}$$

$$\Sigma(s) = \frac{g^2_{K\bar{K}}}{16\pi}\left\{i\rho + \frac{1}{\pi}\left[2 - \rho \ln\frac{1+\rho}{1-\rho}\right]\right\}, \tag{15}$$

where we have returned to the dimensionfull coupling constant $g^2_{K\bar{K}}$, and where $\rho = 2k/\sqrt{s} \simeq k/m$. The contribution of the pion loop has been omitted. The nonrelativistic reduction of (14) -(15) reads

$$D(E) \simeq -2m\bar{g}_{K\bar{K}}\left\{\left(\frac{2}{\bar{g}_{K\bar{K}}}E_n - \frac{2m}{\pi}\right) + \frac{1}{2}\left(-\frac{4}{\bar{g}m} + \frac{4}{\pi m}\right)k^2 - ik\right\}. \tag{16}$$

From (16) one obtains

$$r'_e = -\frac{4}{\bar{g}_{K\bar{K}}m}\left(1 - \frac{\bar{g}_{K\bar{K}}}{\pi}\right) = r_e\left(1 - \frac{\bar{g}_{K\bar{K}}}{\pi}\right). \tag{17}$$

We have recovered the same structure of r'_e as the one which is given by the model (8). The additional term $\bar{g}_{K\bar{K}}/\pi$ varies from 0.4 to 0.9 for the values of $\bar{g}_{K\bar{K}}$ from the Table 2 of Ref.[2]. Comparing (12) and (17) we conclude that they are equivalent provided $b = 3/\pi m$ which corresponds to b_{min} (see the text after (12)). This is not surprising since the only scale parameter with the dimension of length in the loop diagram is $1/m$. Physically more sensible estimate is $b \simeq M^{1/3}$ GeV$^{-1} \simeq 1$ fm [4, 14, 15, 16].

Finally we wish to note that Coulomb effects in K^+K^- system have been neglected. They become really important for $k \lesssim 2\pi/a_B = \pi\alpha m$, where a_B is the Bohr radius of the K^+K^- atom and $\alpha = 1/137$. Therefore the

expression (2) for the S-matrix has to be modified in the energy interval $E \lesssim 0.3$ MeV. The correct form of the f_0 -meson propagator with Coulomb interaction included was derived in Ref. [23], and in Ref [17] the interplay of the f_0 -meson and K^+K^- atom was described.

We have shown that the expression (1) for the effective range undergoes a substantial change due to the term proportional to k^2 arising from $\varepsilon_{K\bar{K}}(k)$. Next task would be to reconsider in a similar way other quantities depending on the T-matrix (5), e.g. the spectral densities of the f_0/a_0 mesons [2], or the positions of the poles [4].

In conclusion, we may repeat the general statement that the problem of the f_0 - meson makeup is far from being resolved. It is possible that more information may be obtained using quantum mechanical approach. In particular the effective range parameter may be an important quantity. The accurate evaluation of this parameter has been given in the present note.

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